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MULTIPLEX BUS DATA WAVEFORM SPECTRA

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This paper contains a derivation of the formula for the average energy spectral density of a multiplex bus data word. It also contains graphs of the spectral density and the cumulative distribution of energy versus frequency.

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SECTION I

INTRODUCTION

MITRE Project 6370, Radio Information Distribution Systems (RIDS), is investigating the use of multiplex busing techniques in aircraft to facilitate the interconnection of avionic subsystems. In August of 1973, the Air Force issued a standard, MIL-STD-1553 (USAF)[1], which is a fundamental reference document for this project. This document defines a set of standards for time division multiplex digital data bus systems. It specifies the bus system configuration, the signal structure, the word format, data rates and the bus interface circuitry.

One aspect of MITRE's work on RIDS is to suggest changes to MIL-STD-1553 where this standard in its present form is unworkable or unduly restrictive. One area in which it was felt that the standard might present unnecessary difficulties to a system designer is that of the input impedance to the Multiplex Terminal Unit (Paragraph 4.2.5.4.4 of MIL-STD-1553). For the frequency range of dc to 100 kHz, this is specified to be at least 6800 ohms. Since the impedance is measured at a pair of terminals which are connected by a transformer winding, it is expected that this specification would be difficult to meet.

It was suspected that because of the way in which the bus waveforms were specified, very little of the waveform energy would be found below 100 kHz. To the extent that this is true, the low frequency input impedance specification could be relaxed without serious consequence. This, then, is the motivation for the study of multiplex bus waveform spectra as described in this paper.

According to the standard, there are three types of words which are transmitted on the bus: command, data and status. Each word consists of a sync waveform and a sequence of seventeen Manchester II bi-phase level waveforms representing sixteen information bits and one parity bit. Since the waveforms on the multiplex bus are sample functions from a random process, their spectral characteristics are best described by a power spectrum. The computation of the power spectrum for the complete waveform process on the bus would require knowledge of the statistics of occurrences and timing of all three types of words and of the information bits within each word. These statistics are system dependent and cannot be deduced from the standard. Thus, we cannot compute this power spectrum, even if we were willing to expend the effort to do so.

Although it is not possible to compute the power spectrum of the complete waveform process on the bus, other types of spectral computations are feasible. The principal criteria for choice among such spectra is that the spectrum chosen be reasonably easy to compute and representative of bus waveform spectra in general. Given these criteria, an appropriate waveform on which to base a spectral computation is the waveform corresponding to a single data word. It is relatively simple compared to the totality of waveforms on the bus, and it is reasonable to assume for this waveform that the underlying information bits are independent and have equal probability of being '0' or '1'. These assumptions are not as reasonable for the command and status words. Since there are 2^{16} possible data words, it will be appropriate to compute the average energy spectral density, where the averaging is done over all possible data words. It will also be necessary to make some assumptions regarding the wave shape of the data word waveforms as these are not completely specified by the standard.

According to MIL-STD-1553, a data word has duration of 20 μs. This duration comprises 20 bit times of 1 μs each. The first three bit times are occupied by the sync waveform, the next sixteen bit times by waveforms representing the sixteen information bits and the last bit time by a waveform representing the parity bit. The elementary waveforms representing information or parity bits are called Manchester II bi-phase level waveforms. The sync waveform is an invalid Manchester waveform (i.e., it could not be the result of a sequence of Manchester waveforms representing information bits). The parity bit is chosen so that the resultant sequence has odd parity. In any information or parity bit position, the waveform representing a '1' is the negative of the waveform representing a '0'. There is a minimum of 150 ns specified for the waveform rise and fall times, when measured at the ten percent and ninety percent points of the specified signal voltage limits.

We shall make the assumption that the waveforms are all trapezoidal or triangular in shape and that the rise and fall times of sync, data, and parity waveforms are all equal. If, in addition, this common rise/fall time is precisely 150 ns as measured at the ten percent and ninety percent points of the waveforms, then the assumed waveshape has 187 ns rise/fall times as measured at the 0 and 100 percent points. Making all these assumptions, the elementary waveforms out of which a data word is formed, are shown in Figure 1. The elementary data or parity waveform shown in the figure represents a '0'.

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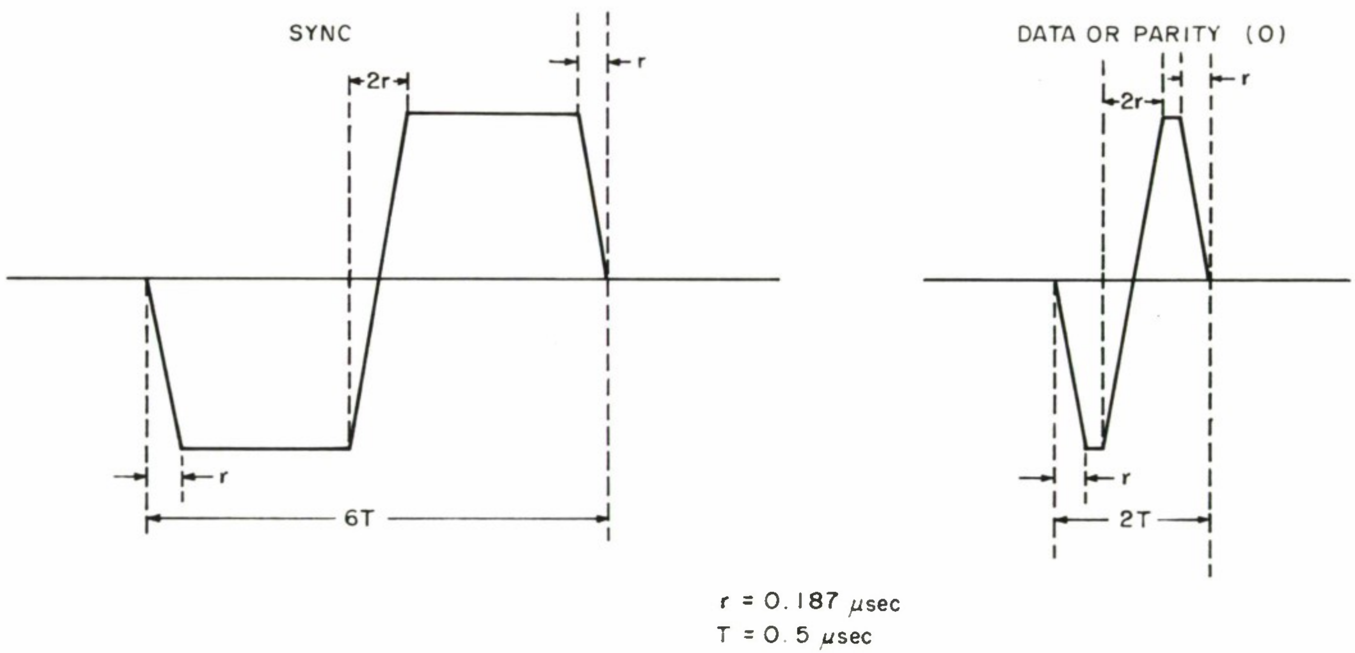

Figure 1 ELEMENTARY WAVEFORMS

SECTION II
FUNDAMENTAL RELATIONSHIPS

If f(t) is a (absolutely integrable) real function of time with Fourier transform $F(\omega)$, defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{1}$$

then we may define

$$\Phi_{ff}(\omega) \triangleq \frac{1}{2\pi} \left| F(\omega) \right|^2 \tag{2}$$

as the <u>energy density spectrum</u> of f(t).

Parseval's Theorem gives us

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left| F(\omega) \right|^2 d\omega \tag{3}$$

Thus, the integral over frequency of the energy density spectrum is the waveform energy, as required.

Let the (time) autocorrelation function, $\phi_{ff}(\tau)$, be given by

$$\phi_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)\, f(t + \tau)\, dt \tag{4}$$

Note that if $g(t) = \pm f(t + s)$, where s is a constant, then $\phi_{gg}(\tau) = \phi_{ff}(\tau)$. Then

$$\left| F(\omega) \right|^2 = \int_{-\infty}^{\infty} \phi_{ff}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{5}$$

Thus,

$$\Phi_{ff}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{ff}(\tau)\ e^{-j\omega\tau}\ d\tau \quad * \tag{6}$$

Let f(t) be a random function of time which is a member of the set A. If 'E' denotes expectation, we may define

$$S_{AA}(\omega) = \underset{f\varepsilon A}{E}\left[\Phi_{ff}(\omega)\right] \tag{7}$$

as an average energy density spectrum. For the situation of interest to us, f is a particular data word and A is the set of all possible data words. We shall defer the derivation of the average energy spectrum of a multiplex bus data word to the Appendix, and here give only the result, $S(\omega)$. This is

$$S(\omega) = \begin{cases} 0 \text{ for } \omega \leq 0 \\ \frac{64}{\pi\omega^4 r^2} \sin^2 \frac{\omega r}{2} \left[\sin^2 \frac{\omega(3T-r)}{2} \sin^2 \frac{3\omega T}{2} + 17 \sin^2 \right. \\ \left. \qquad \frac{\omega(T-r)}{2} \sin^2 \frac{\omega T}{2}\right] \\ \qquad\qquad \text{for } 0 < \omega < \infty \end{cases} \tag{8}$$

This is a single sided (positive frequencies only) spectrum. The formula is valid for $0 < r \leq 0.5T$. At the upper limit for r, the waveforms representing data bits become triangular. The average

---

* The development, up to this point, is basically similar to that in Reference 2, pp. 36-45. Thus, we have omitted all proofs.

energy spectral density with respect to cycle frequency, f, rather than radian frequency, ω, may be obtained by multiplying the expression for $S(\omega)$ above by 2π, and substituting $2\pi f$ for ω. The average energy spectral density is plotted in Figures 2 and 3 for the two limiting cases r = 0.187 μsec. and r = 0.25 μsec. Note that, in either case, the bulk of the spectral energy is found between 0 and 2 MHz. When r = 0.187 μsec., 98.7 percent of the energy is found between 0 and 2 MHz. For r = 0.25 μsec., the comparable figure is 98.0 percent. Appreciable additional energy is found between 2 MHz and 3 MHz for r = 0.187 μsec. (1.1 percent) and between 2 MHz and 3.26 MHz for r = 0.25 μsec. (1.8 percent).

Of particular interest is the proportion of energy at frequencies less than 100 kHz. This comprises 0.95 percent of the total when r = 0.187 μsec. and 1.16 percent of the total when r = 0.25 μsec. More complete information about low frequency content is given by the cumulative distribution of energy for the range 0 to 100 kHz. This is plotted in Figures 4 and 5 for the cases r = 0.187 μsec. and r = 0.25 μsec., respectively. Of some interest also is the cumulative distribution of energy over the broad range of frequencies which contain the bulk of the spectral energy. We shall consider this range to be 0 to 4 MHz. These broad range cumulative distributions are plotted in Figures 6 and 7 for the cases r = 0.187 μsec. and r = 0.25 μsec., respectively.

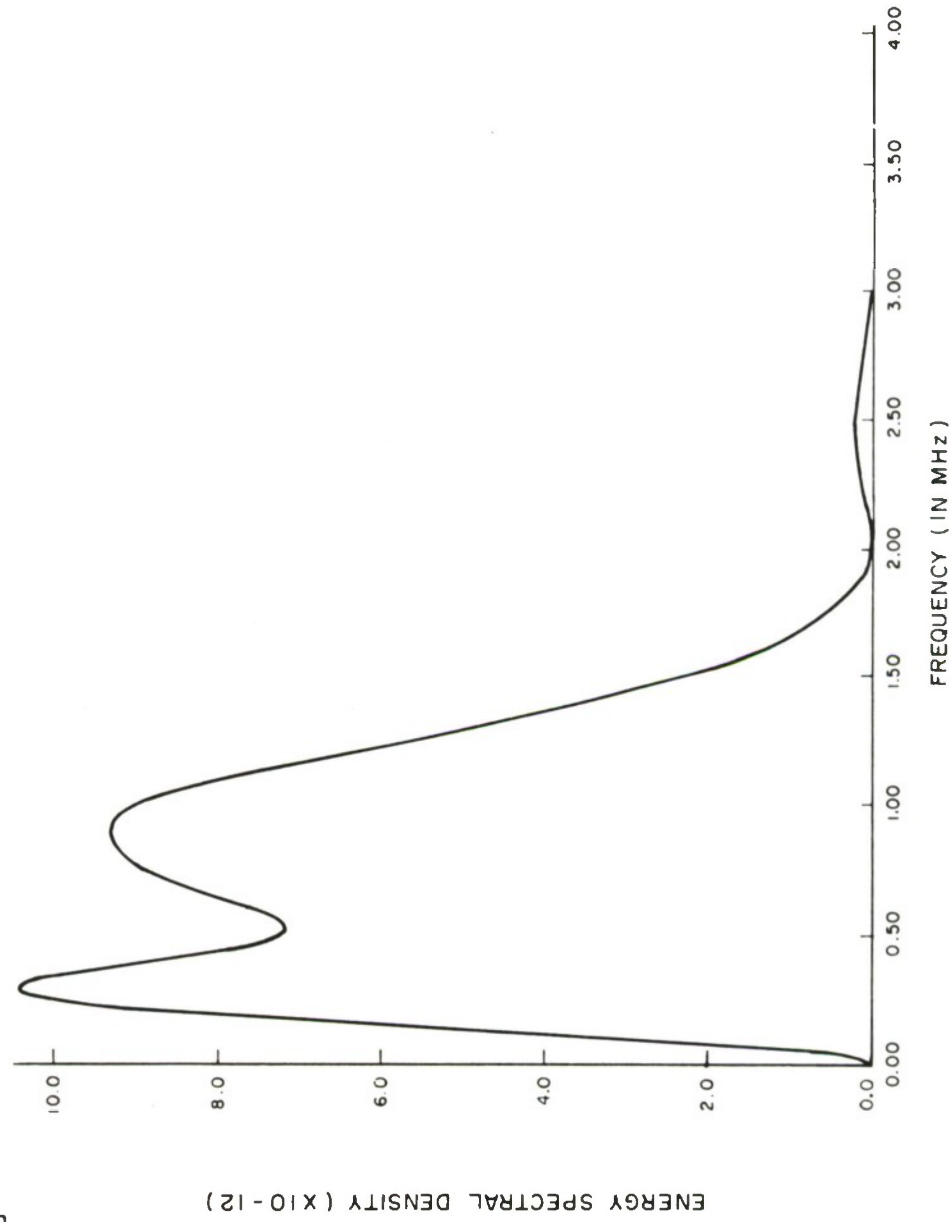

Figure 2 AVERAGE ENERGY SPECTRAL DENSITY , r = 0.187 MICROSECOND

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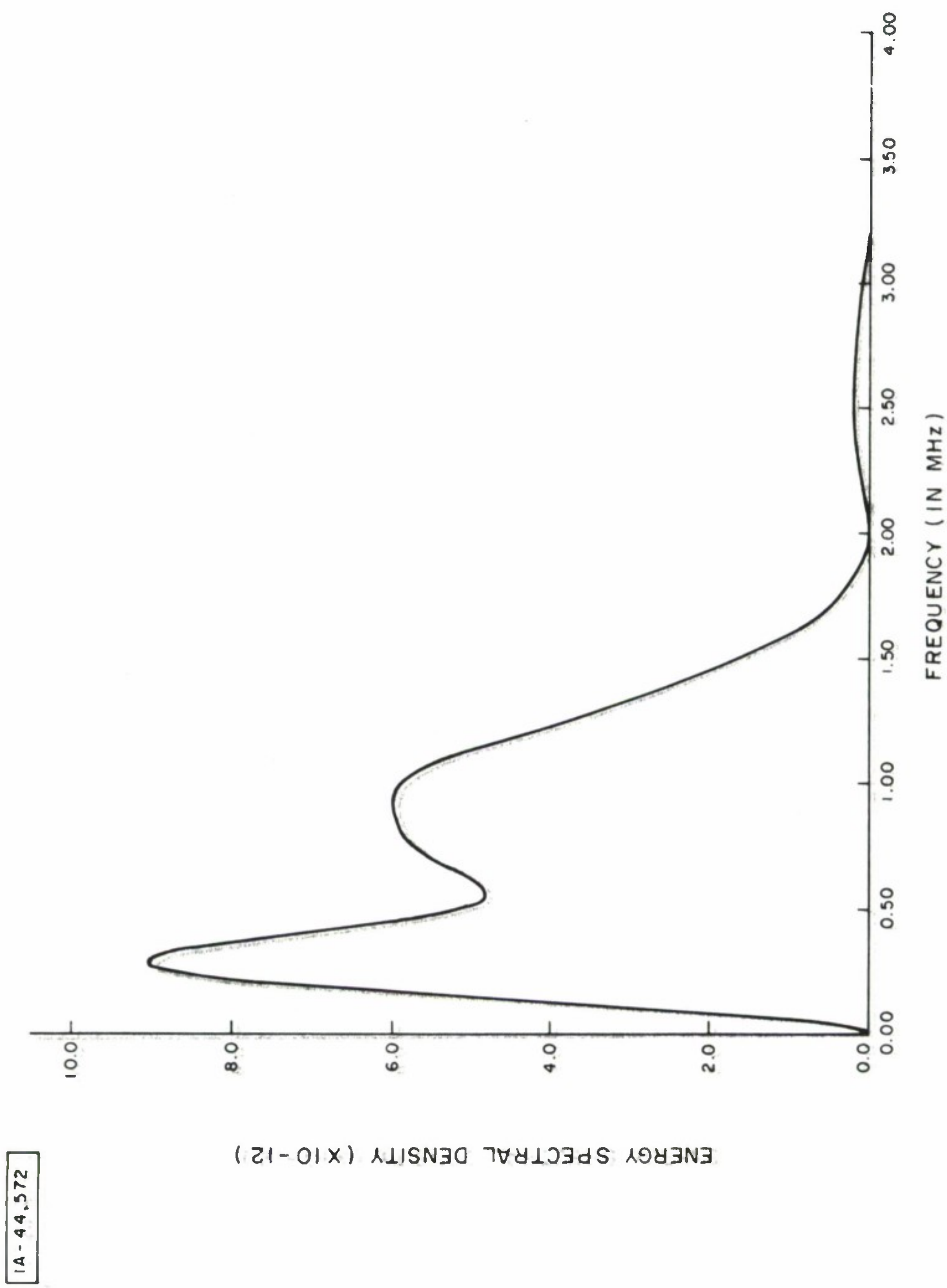

Figure 3 AVERAGE ENERGY SPECTRAL DENSITY, r = 0.25 MICROSECOND

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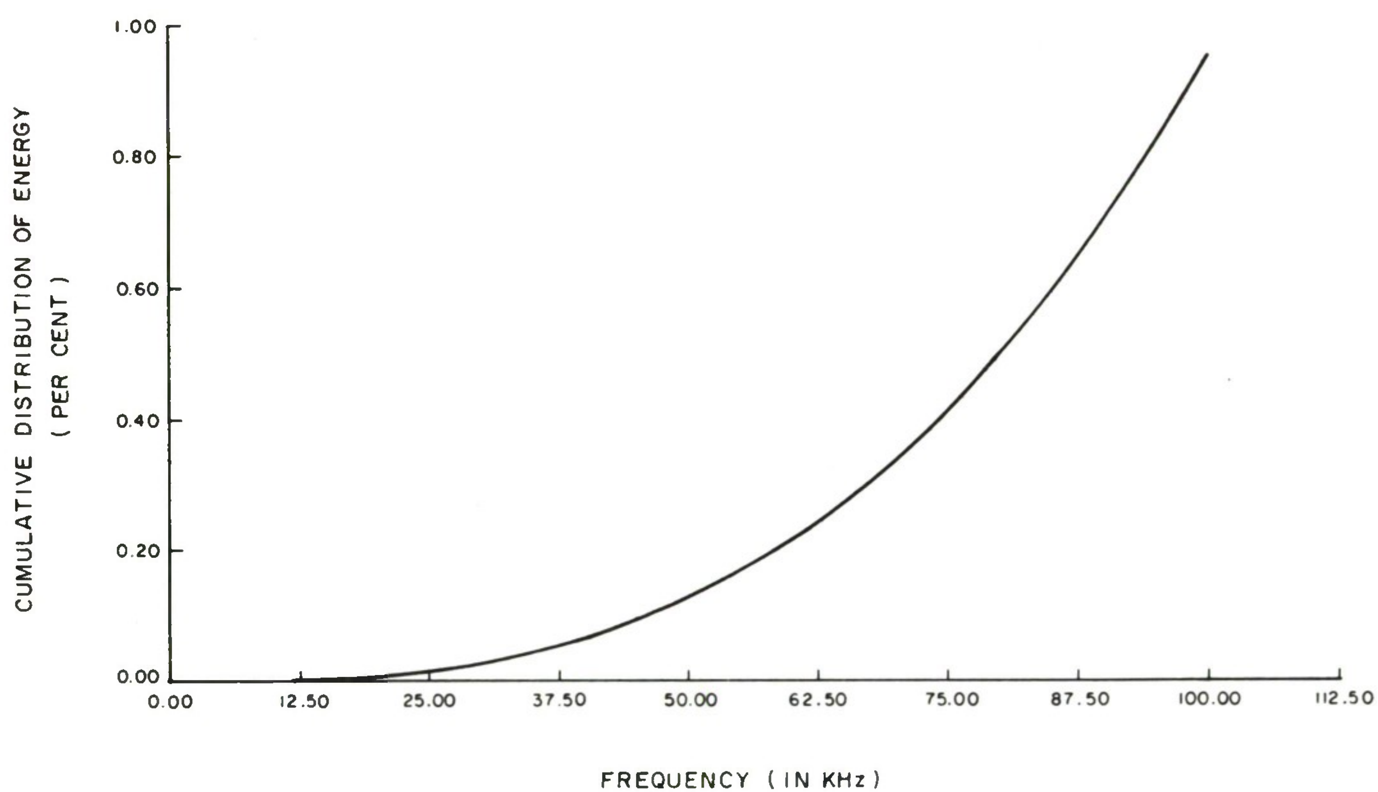

Figure 4 CUMULATIVE DISTRIBUTION OF ENERGY AT LOW FREQUENCIES , r = 0.187 MICROSECOND

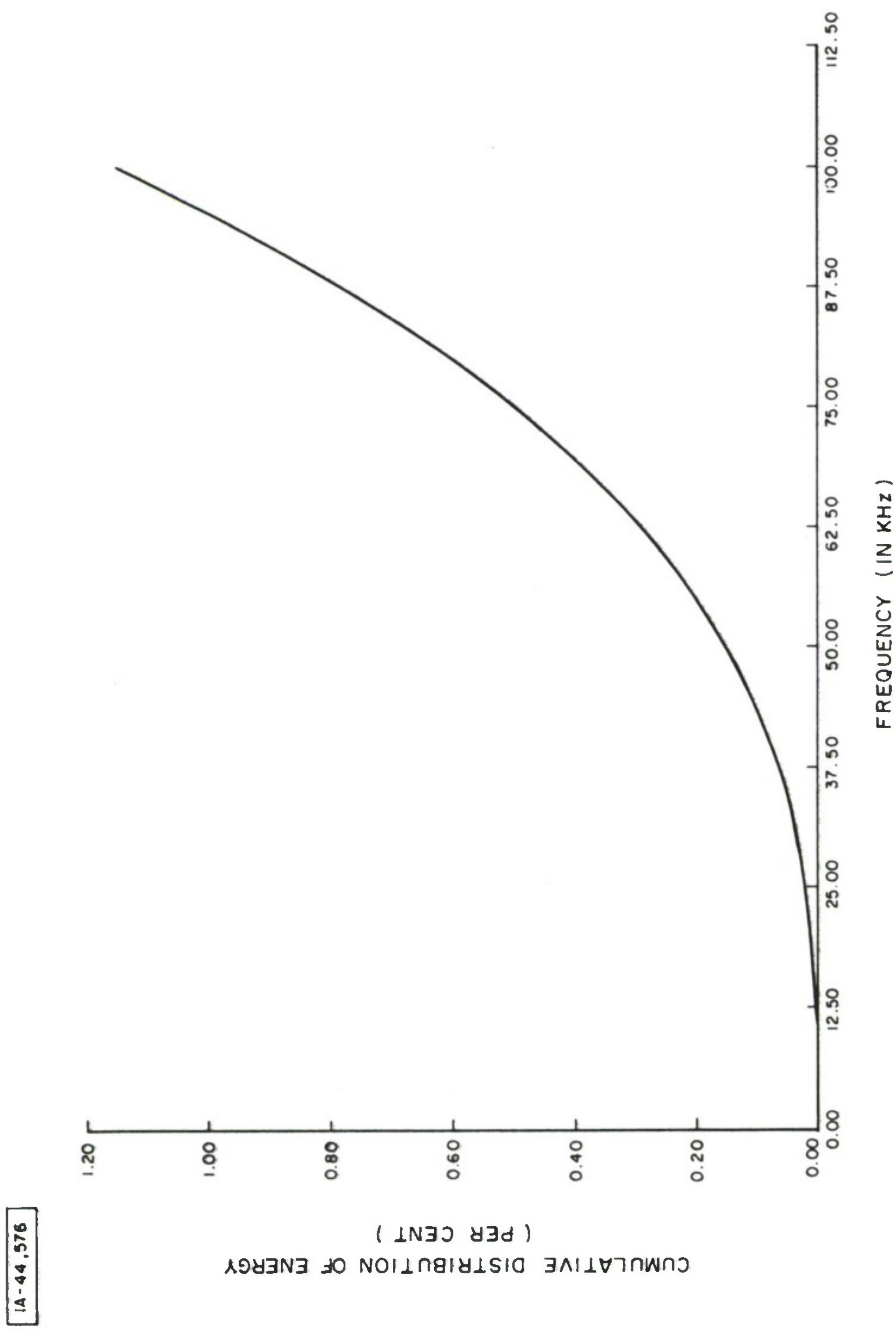

Figure 5 CUMULATIVE DISTRIBUTION OF ENERGY AT LOW FREQUENCIES , r = 0.25 MICROSECOND

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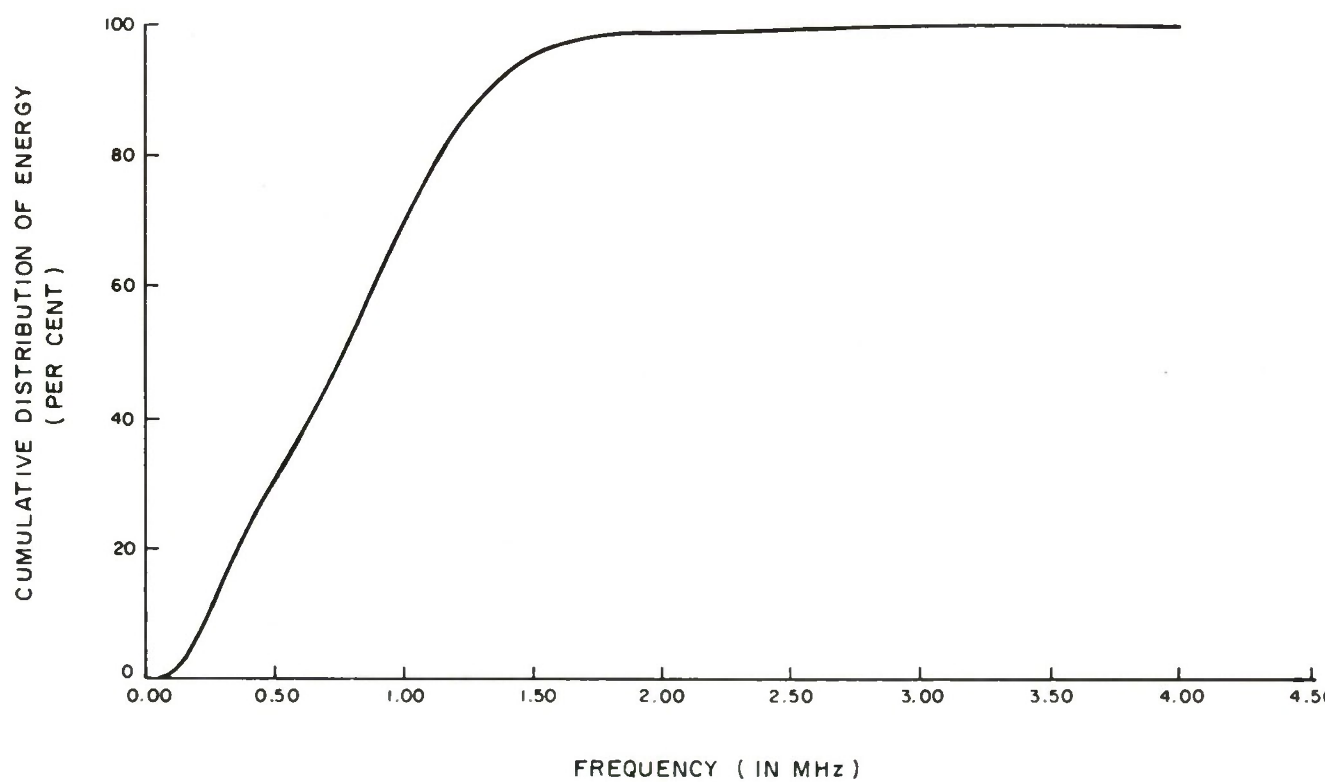

Figure 6 CUMULATIVE DISTRIBUTION OF ENERGY , r = 0.187 MICROSECOND

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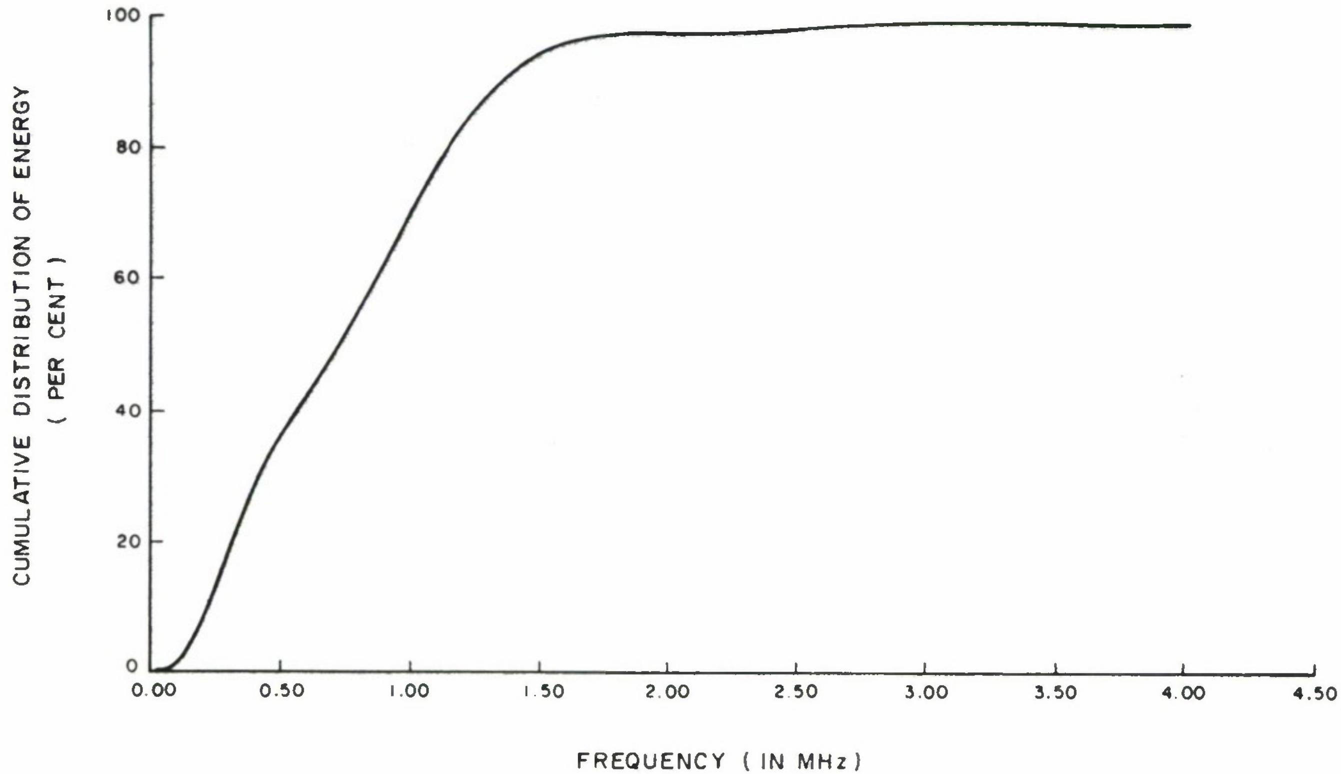

Figure 7 CUMULATIVE DISTRIBUTION OF ENERGY , r = 0.25 MICROSECOND

SECTION III
CONCLUSIONS

This study was motivated by a suspicion that perhaps the low frequency Multiplex Terminal Unit (MTU) input impedance specifications in MIL-STD-1553 were unnecessarily restrictive. The results of the study support this notion. Since only about one percent of the signal energy lies below 100 kHz, the effect of the MTU input impedance in that frequency range on the bus waveform should be similarly small. Thus, a lowering of the present 6800 ohm minimum impedance specification for the range 0 to 100 kHz to a value or values more easily realizable by the specified MTU input circuit configuration is indicated.

It should, however, be realized that we have not provided a conclusive theoretical proof that the energy below 100 kHz cannot appreciably effect synchronization or detectability. There are two major missing ingredients for such a proof. First, we have only considered an average energy spectrum. In the analysis of changes in detectability, fluctuations about this average may be important, as the probability of a particular "worst case" data waveform (2^{-16}) is large compared to the sort of error probabilities we are interested in achieving (approximately 10^{-7}). Second, it might be argued that the frequency content below 100 kHz has a disproportionately large influence upon detectability.

In fact, rudimentary analysis indicates that the "worst case" waveform is not significantly worse than the average we have computed at the low frequency end of the spectrum. Further, it seems unlikely, in view of the very great distortion of the waveform by the bus, that small amounts of energy in any part of the spectrum could effect detectability in a significantly disproportionate way. It would,

however, require a quite substantial analytical effort to prove that both these beliefs are correct.

Fortunately, experimental evidence supporting the notion that the omission of low frequencies does not seriously effect synchronization or detectability has recently been obtained by T. Allen. Allen's work* involved high-pass filtering the data waveform before detection. His results show that for filter cut-off frequencies below 100 kHz, synchronization and detection may be accomplished very nearly as well as when no filter is present. This was found to be the case despite the fact that the filters used caused significant distortion in regions of the spectrum above the cut-off frequency.

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* To be published.

APPENDIX
DERIVATION OF THE EXPRESSION FOR THE AVERAGE ENERGY SPECTRUM OF THE MULTIPLEX BUS DATA WAVEFORM

Using our equations (6), (7) and (4) we may obtain

$$
\begin{aligned}
S_{AA}(\omega) &= \mathop{E}_{f\varepsilon A}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{ff}(\tau)\, e^{-j\omega\tau}\, d\tau\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\mathop{E}_{f\varepsilon A}\ \phi_{ff}(\tau)\right] e^{-j\omega\tau}\, d\tau \qquad \text{(A1)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty} \mathop{E}_{f\varepsilon A}\left[f(t)\ f(t+\tau)\right] dt\, d\tau
\end{aligned}
$$

We shall define

$$
\begin{aligned}
R_{AA}(\tau) &\triangleq \mathop{E}_{f\varepsilon A}\left[\phi_{ff}(\tau)\right] \\
&= \mathop{E}_{f\varepsilon A}\int_{-\infty}^{\infty} f(t)\ f(t+\tau)\ dt \qquad \text{(A2)} \\
&= \int_{-\infty}^{\infty} \mathop{E}_{f\varepsilon A}\left[f(t)\ f(t+\tau)\right] dt
\end{aligned}
$$

Hence, from (A1) and (A2),

$$
S_{AA}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{AA}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{(A3)}
$$

Note that if we define $f^d(t)$ by

$$f^d(t) \triangleq f\ (t + d) \tag{A4}$$

for some constant, d, and all $f \epsilon A$, then

$$R^d{}_{AA}(\tau) = R_{AA}(\tau) \tag{A5}$$

and

$$S^d{}_{AA}(\omega) = S_{AA}(\omega) \tag{A6}$$

Thus, the average autocorrelation and average energy spectrum are unaffected by a shift in the time origin. If, for real functions of time, f(t), g(t), we define a (time) crosscorrelation, $\phi_{fg}(\tau)$ as

$$\phi_{fg}(\tau) \triangleq \int_{-\infty}^{\infty} f(t)\ g(t + \tau)\ dt \tag{A7}$$

we have

$$\phi_{fg}(\tau) = \phi_{gf}(-\tau) \tag{A8}$$

For f(t), $f \epsilon A$, g(t), $g \epsilon B$, we may define

$$\begin{aligned} R_{AB}(\tau) &= \underset{\substack{f \epsilon A \\ g \epsilon B}}{E}\ \phi_{fg}(\tau) \\ &= \underset{\substack{f \epsilon A \\ g \epsilon B}}{E} \int_{-\infty}^{\infty} f(t)\ g(t + \tau)\ dt \end{aligned}$$

$$= \int_{-\infty}^{\infty} \underset{\substack{f \varepsilon A \\ g \varepsilon B}}{E} \left[f(t)\ g\ (t + \tau) \right] dt \qquad \text{(A9)}$$

Assuming the waveform time origin to be fixed, let D be the set consisting of the 2^{16} possible information waveforms (i.e., waveforms occupying bit times 4 through 19), let P be the set consisting of the two possible parity waveforms, let Y be the set consisting of the one sync waveform appropriate to a data word, and let W be the set consisting of the 2^{16} possible (entire) data word waveforms. Let a member of the sets D, P, Y and W be denoted by d(t), p(t), y(t) and w(t), respectively.

Then we have

$$w(t) = y(t) + d(t) + p(t) \qquad \text{(A10)}$$

Using (4), (A2), (A9), and the fact that the expected value of a sum of random variables is equal to the sum of the expected values, we have

$$R_{WW}(\tau) = R_{YY}(\tau) + R_{YD}(\tau) + R_{YP}(\tau) + R_{DY}(\tau) + R_{DD}(\tau)$$

$$+ R_{DP}(\tau) + R_{PY}(\tau) + R_{PD}(\tau) + R_{PP}(\tau) \qquad \text{(A11)}$$

Let $h_k(t)$ be the waveform representing a '0' in the k^{th} information bit ($k = 1, \ldots., 16$). Let $h_{17}(t)$ be the waveform representing a '0' in the parity bit. Then, $-h_k(t)$ is the waveform representing a '1' in the k^{th} information bit and $-h_{17}(t)$ is the waveform representing a '1' in the parity bit. We shall let b_k, $k = 1, \ldots., 16$, be the k^{th} information bit, and b_{17} be the parity bit. We shall denote the probability of occurrence of an event V by Pr {V}.

Lemma 1

$$\mathop{E}_{d \epsilon D} \left[d(t) \right] = 0 \quad \text{all } t \tag{A12}$$

Proof

If t does not lie within bit times 4 through 19, then $d(t) = 0$. If for some k, $k = 1, \ldots, 16$, t lies within bit time $k + 3$, then

$$\mathop{E}_{d \epsilon D} \left[d(t) \right] = Pr\{b_k = 0\} \cdot h_k(t) + P\{b_k = 1\} \cdot \left[-h_k(t) \right]$$

$$= 0$$

since $Pr\{b_k = 0\} = \frac{1}{2} = Pr\{b_k = 1\}$, by assumption.

Lemma 2

$$R_{YD}(\tau) = 0, \text{ all } \tau \tag{A13}$$

$$R_{DY}(\tau) = 0, \text{ all } \tau \tag{A14}$$

From (A9),

$$R_{YD}(\tau) = \mathop{E}_{\substack{y \epsilon Y \\ d \epsilon D}} \int_{-\infty}^{\infty} y(t)\ d\ (t + \tau)\ dt$$

$$= \int_{-\infty}^{\infty} y(t) \mathop{E}_{d \epsilon D} \left[d(t + \tau) \right] dt$$

$$= 0 \quad \text{by (A12)}$$

Then we may obtain (A14) from (A8), (A9), and (A13).

Lemma 3

$$\Pr\{b_{17} = 0\} = \Pr\{b_{17} = 1\} = \tfrac{1}{2} \tag{A15}$$

Proof

Note that for M a positive integer,

$$\sum_{k=0}^{M} \binom{M}{k} (-1)^k = (x+y)^M \Big|_{\substack{x=-1 \\ y=1}} = 0 \tag{A16}$$

Since b_{17} is chosen to give odd parity,

$$\Pr\{b_{17} = 1\} = \sum_{\substack{k=0 \\ k \text{ even}}}^{16} \binom{16}{k} (\tfrac{1}{2})^{16} = \sum_{\substack{k=0 \\ k \text{ even}}}^{16} \binom{16}{k} (-1)^k (\tfrac{1}{2})^{16} \tag{A17}$$

$$\Pr\{b_{17} = 0\} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{16} \binom{16}{k} (\tfrac{1}{2})^{16} = -\sum_{\substack{k=0 \\ k \text{ odd}}}^{16} \binom{16}{k} (-1)^k (\tfrac{1}{2})^{16} \tag{A18}$$

Subtracting (A18) from (A17) we have by (A16) that

$$\Pr\{b_{17} = 1\} - \Pr\{b_{17} = 0\} = 0 \tag{A19}$$

Since '0' and '1' are the only possible values for b_{17},

$$\Pr\{b_{17} = 1\} + \Pr\{b_{17} = 0\} = 1 \qquad \text{(A20}$$

and combining (A19) with (A20) gives us (A15).

Lemma 4

$$R_{YP}(\tau) = 0 \quad \text{all } \tau \tag{A21}$$

$$R_{PY}(\tau) = 0 \quad \text{all } \tau \tag{A22}$$

Proof

The result is a consequence of (A15). The proof follows that of Lemmas 1 and 2.

Lemma 5

$$R_{DP}(\tau) = 0 \quad \text{all } \tau \tag{A23}$$

$$R_{PD}(\tau) = 0 \quad \text{all } \tau \tag{A24}$$

Proof

Pick arbitrary values of t and τ. If t lies outside bit times 4 through 19, then $d(t) = 0$ irrespective of the values of the information and parity bits, $\{b_k\}_{k=1}^{17}$. Thus

$$\underset{\substack{d\varepsilon D \\ p\varepsilon P}}{E}\left[d(t)\ p(t+\tau)\right] = 0 \tag{A25}$$

for t outside bit times 4 through 19 and all τ.

Suppose t lies in the $(k+3)^{rd}$ bit time for some $1 \leq k \leq 16$. Then

$$\underset{\substack{d\varepsilon D \\ p\varepsilon P}}{E}\left[d(t)\ p(t+\tau)\right] = \underset{b_k,\ b_{17}}{E}\left[(-1)^{b_k}\ h_k(t)\cdot(-1)^{b_{17}}\ h_{17}(t+\tau)\right]$$

$$= \underset{b_k,\ b_{17}}{E}\left[(-1)^{b_k + b_{17}}\right] h_k(t)\ h_{17}(t+\tau) \tag{A26}$$

Let

$$\{b_\ell\}' = \{b_\ell\}_{\substack{\ell=1 \\ \ell\neq k}}^{16} \tag{A27}$$

b_{17} is chosen to produce odd parity in the 17 bit information and parity bit sequence. Thus, if $b_k = 0$, $b_{17} = 0$ when $\{b_\ell\}'$ has odd parity. If $b_k = 1$, $b_{17} = 1$ when $\{b_\ell\}'$ has odd parity.

$$\Pr\{b_k + b_{17} = 0 \text{ or } b_k + b_{17} = 2\}$$

$$= \Pr\{b_k = 0\}\ \Pr\{\{b\ell\}' \text{ odd}\} + \Pr\{b_k = 1\}\ \Pr\{\{b_\ell\}' \text{ odd}\}$$

$$= \Pr\{\{b_\ell\}' \text{ odd}\}$$

$$= \tfrac{1}{2} \tag{A28}$$

where the last step may be obtained by an argument similar to that used to prove Lemma 3. (Set M = 15 in (A16)). Since $b_k + b_{17} = 1$ is the only possibility not covered in the left hand side of (A28),

$$\Pr\{b_k + b_{17} = 1\} = \tfrac{1}{2} \tag{A29}$$

Thus, from (A26), (A28), and (A29)

$$\underset{\substack{d \epsilon D \\ p \epsilon P}}{E} \left[d(t)\ p(t + \tau) \right] = 0 \quad \text{for } t \text{ within bit times 4 through 19 and all } \tau \tag{A30}$$

Taken together, (A25) and (A30) constitute a statement that

$$\underset{\substack{d \epsilon D \\ p \epsilon P}}{E} \left[d(t)\ p(t + \tau) \right] = 0 \quad \text{all } t \text{ and } \tau \tag{A31}$$

(A23) and (A24) follow from (A9) and (A8).

Combining (A11), (A13), (A14), (A21), (A22), (A23), and (A24), we obtain

$$R_{WW}(\tau) = R_{YY}(\tau) + R_{DD}(\tau) + R_{PP}(\tau) \tag{A32}$$

Now

$$d(t) = \sum_{k=1}^{16} (-1)^{b_k}\ h_k(t) \tag{A33}$$

and

$$\begin{aligned} E\left[d(t)\ d(t + \tau)\right] &= E\left[\sum_{\ell=1}^{16} \sum_{k=1}^{16} (-1)^{b_k} \cdot (-1)^{b_\ell}\ h_k(t)\ h_\ell(t + \tau)\right] \\ &= \sum_{\ell=1}^{16} \sum_{k=1}^{16} E\left[(-1)^{b_k} \cdot (-1)^{b_\ell}\right] h_k(t)\ h_\ell(t + \tau) \end{aligned} \tag{A34}$$

If $k \neq \ell$,

$$E\left[(-1)^{b_k} \cdot (-1)^{b_\ell}\right] = E\left[(-1)^{b_k}\right] \cdot E\left[(-1)^{b_\ell}\right]$$

$$= 0 \tag{A35}$$

since the b_k's are independent and have probability ½ of being either 0 or 1. If $k = \ell$,

$$E\left[(-1)^{2b_k}\right] = 1 \tag{A36}$$

Since $2\,b_k$ is either 0 or 2 and

$$(-1)^o = (-1)^2 = 1 \tag{A37}$$

Thus,

$$E\left[d(t)\ d(t + \tau)\right] = \sum_{k=1}^{16} h_k(t)\ h_k(t + \tau) \tag{A38}$$

Now each function $h_k(t)$ is merely a time shifted version of $h_1(t)$. The autocorrelation is unaffected by such time shifts, by the remark following (4). Thus, if we define

$$\phi_{hh}(\tau) \triangleq \int_{-\infty}^{\infty} h_1(t)\ h_1(t + \tau)\ dt \tag{A39}$$

we may combine (A2), (A38), and (A39) to get

$$R_{DD}(\tau) = 16\ \phi_{hh}(\tau) \tag{A40}$$

Now,

$$E\left[p(t)\ p(t+\tau)\right] = E\left[(-1)^{b_{17}} \cdot (-1)^{b_{17}}\right] h_{17}(t)\ h_{17}(t+\tau)$$

$$= h_{17}(t)\ h_{17}(t+\tau) \tag{A41}$$

by (A37). Since h_{17} is a time shifted version of h_1,

$$R_{PP}(\tau) = \phi_{hh}(\tau) \tag{A42}$$

Since the sync waveform, $y(t)$, is deterministic, we may state the equivalence of the notations for its average autocorrelation and deterministic waveform autocorrelation, i.e.,

$$R_{YY}(\tau) = \phi_{yy}(\tau) \tag{A43}$$

Thus from (A32), (A40), (A42), and (A43) we get

$$R_{WW}(\tau) = \phi_{yy}(\tau) + 17\ \phi_{hh}(\tau) \tag{A44}$$

By Fourier transforming (A44), we get

$$S_{WW}(\omega) = \Phi_{yy}(\omega) + 17\ \Phi_{hh}(\omega) \tag{A45}$$

What we have done thus far is to reduce the expression for the average autocorrelation of a complicated random function to a weighted sum of simpler (deterministic) waveform autocorrelations. This has been accomplished using the nature of the waveform representation (i.e., the waveform representing a '1' in a particular information or parity bit is the negative of the waveform representing a zero in the same bit), the probability distribution on the information bits (i.e., independent bits, equiprobably '0' or '1'), the odd parity require-

ment, and the fact that the sync waveform is perfectly deterministic (i.e., independent of the information bits in a data word). The nature of the waveforms representing the sync and a '0' information or parity bit has not entered into the analysis. We shall now take advantage of the nature of these waveforms to achieve a further simplification.

$y(t)$ and $h_1(t)$ both have the property of odd symmetry about their mid-points. Waveforms with this property have autocorrelation functions and energy spectra which are derivable from those of that portion of the original waveform which lies on one side of the mid-point.

Lemma 6

Let

$$v(t) = u(t) - u(t + Q) \tag{A46}$$

where $u(t)$ is a waveform which is zero outside the interval $(0,Q)$. Let $U(\omega)$ be the Fourier transform of $u(t)$. Then

$$\phi_{vv}(\tau) = 2\,\phi_{uu}(\tau) - \phi_{uu}(\tau - Q) - \phi_{uu}(\tau + Q) \tag{A47}$$

and

$$\Phi_{vv}(\omega) = \frac{2}{\pi}\,|U(\omega)|^2 \sin^2 \frac{\omega Q}{2} \tag{A48}$$

Proof

$$\phi_{vv}(\tau) = \int_{-\infty}^{\infty} [u(t) - u(t + Q)]\,[u(t + \tau) - u(t + \tau + Q)]\,dt$$

$$= 2\ \phi_{uu}(\tau) - \phi_{uu}(\tau - Q) - \phi_{uu}(\tau + Q) \qquad \text{(A49)}$$

Taking the Fourier transform of (A49) we get

$$\Phi_{vv}(\omega) = 2\ \Phi_{uu}(\omega)\ (1 - \cos \omega Q) \qquad \text{(A50)}$$

Using (2) and a trigonometric identity we get (A48) from (A50).

Both $h_1(t)$ and $y(t)$ may be decomposed as per equation (A46). For $h_1(t)$, Q = ½ bit time = T. For $y(t)$, $Q = 3T$. (See Figure 1.) $u(t)$ represents the positive part of each waveform.

Let $b(t)$ be the positive (right) half of $h_1(t)$ and $c(t)$ be the positive (right) half of $y(t)$. Let $B(\omega)$ and $C(\omega)$ be the Fourier Transform of $b(t)$ and $c(t)$, respectively. Then, from (A45) and (A48),

$$S_{WW}(\omega) = \frac{2}{\pi} \left[|C(\omega)|^2 \sin^2 \frac{3\omega T}{2} + 17\ |B(\omega)|^2 \sin^2 \frac{\omega T}{2} \right] \qquad \text{(A51)}$$

We have immediately from (A51) that

$$S_{WW}(0) = 0 \qquad \text{(A52)}$$

To get the explicit expression for $S_{WW}(\omega)$, we need the Fourier transform of a trapezoidal waveshape. The easiest way to derive this is to differentiate the waveshape twice to get four impulses, transform the impulses and multiply the transform by $-1/\omega^2$. The result, for a trapezoid $u(t)$ with base Q and rise and fall times both r is:

$$U(\omega) = \frac{4}{\omega^2 r} \sin \omega \frac{(Q-r)}{2} \sin \frac{\omega r}{2} \qquad \text{(A53)}$$

Setting $Q = 3T$ in (A53) gives the expression for $C(\omega)$. Setting $Q = T$

gives the expression for $B(\omega)$. Combining (A53) and (A51) in this way we get

$$S_{WW}(\omega) = \frac{32}{\pi\omega^4 r^2} \sin^2 \frac{\omega r}{2} \left[\sin^2 \frac{\omega(3T-r)}{2} \sin^2 \frac{3\omega T}{2} + 17 \sin^2 \frac{\omega(T-r)}{2} \sin^2 \frac{\omega T}{2}\right] \tag{A54}$$

The above expression is a double sided spectrum (positive and negative frequencies). The single sided spectrum (8) is obtained from (A54) by doubling the right hand side of (A54) when $\omega > 0$, and setting it to zero when $\omega \leq 0$. Use is made here of the fact that

$$\lim_{\omega \to 0} S_{WW}(\omega) = 0.$$

REFERENCES

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